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Berry Connections and Induced Gauge Fields in Quantum Mechanics on Sphere

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Abstract

Quantum mechanics on sphere S^n is studied from the viewpoint that the Berry's connection has to appear as a topological term in the effective action. Furthermore we show that this term is the Chern-Simons term of gauge variables that correspond to the extra degrees of freedom of the enlarged space.

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1 Introduction

In recent years quantum mechanics on sphere S^n have been studied in various aspects[1][2]. Interestingly, the gauge fields are seen to emerge at the quantum level, which in turn specify the possible inequivalent quantizations and they are coupled with a particle constrained to move on the sphere. In ref.[3] we have presented a picture where the wave functions are constrained to the sphere by the "quantum constraint". Namely we utilize, à la Dirac, a square root of the usual on sphere constraint. In this approach, the induced gauge fields appear in the Hopf map from the wave function space to the real space.

In this note we show the following :

1. By considering the path integral quantization on S^2 through the use of the wave functions mentioned above, we show that the Berry's connection appears as a topological term in the effective action.

2. The topological term is nothing but the Chern-Simons term of gauge variables, which appear when we enlarge our space S^2 to $SO(3)$, by the use of the relation $S^2 = SO(3)/SO(2)$. The gauge variables absorb the extra degrees of freedom of the enlarged space.

3. We generalize the arguments in 1. and 2. to arbitrary sphere S^n.

We recapitulate in section 2 what we have done in the previous paper[3]. Then using the wave function introduced in section 2 we perform the path integral quantization and obtain the effective action on S^2 in section 3. We shall be concerned with the cases $n = 3, 4$ in section 4. In section 5 we consider the general case. Section 6 is devoted to discussions.

2 Hopf Map and Quantization on a Sphere

In the previous paper [3] we have quantized a system constrained to move on a sphere by considering a square root of the "on sphere condition" and have arrived at the fibre bundle structure of the Hopf map in the cases of S^2 and S^4. This leads to more geometrical understanding of monopole and instanton gauge structures that emerge in the course of quantization. We have seen that square root of the "on sphere condition" $x_M^2 - r^2 = 0$ can be written as

$$(x_M \Gamma_M - r)\, \Phi\,(\vec{x}) = 0\ , \tag{1}$$

where Γ_M are the Pauli matrices for S^2 and are the Dirac matrices for S^4 . A solution of eq.(1) is expressed as

$$\Phi = v\phi \ , \tag{2}$$

with ϕ an arbitrary complex function on the sphere. The solution to the constraint is the space projected by P

$$P\Phi = \Phi \ , \tag{3}$$

where the projection operator P to the space spanned by v is defined as

$$\begin{aligned} P &\equiv vv^\dagger \ , \\ P^2 &= vv^\dagger vv^\dagger = P \ , \\ Pv &= v \ . \end{aligned} \tag{4}$$

The explicit form of v can be written as

$$v = \frac{1}{\sqrt{2r\left(r + x_3\right)}} \begin{pmatrix} r + x_3 \\ x_1 + ix_2 \end{pmatrix} \ , \tag{5}$$

for S^2 and

$$v = \frac{1}{\sqrt{2r\left(r + x_5\right)}} \begin{pmatrix} r + x_5 \\ x_4 - i\vec{x}\cdot\vec{\sigma} \end{pmatrix} \ , \tag{6}$$

for S^4 . Then evaluation of the projected derivative $P\partial\Phi$, through

$$\begin{aligned} P\partial\Phi &= vv^\dagger \partial\left(v\phi\right) \\ &= vv^\dagger\left(v\partial\phi + \partial v\phi\right) \\ &= vD\phi \ , \end{aligned} \tag{7}$$

where

$$\begin{aligned} D &\equiv \partial + A \ , \\ A &\equiv v^\dagger \partial v \ , \end{aligned} \tag{8}$$

leads to the induced magnetic monopole gauge potential for S^2

$$A \equiv v^\dagger dv = \frac{i}{2r\left(r + x_3\right)}\left(x_1 dx_2 - x_2 dx_1\right) \ , \tag{9}$$

which was obtained in refs.[1],[2], and the instanton gauge potential for S^4

$$A = i\frac{1}{2r\left(r + x_5\right)}\sigma_{\mu\nu} x_\mu dx_\nu \ , \tag{10}$$

discussed in refs.[4],[5].

3 Path Integral Quantization on S^2

According to the arguments of the previous section, the wave function on S^2 can be written as

$$\Phi = v\phi \ . \tag{11}$$

Using this wave function we calculate the transition amplitude

$$\begin{aligned} Z &\equiv \langle\vec{x}, v; t_f|\vec{x}, v; t_i\rangle \\ &= {}_f\langle\vec{x}, v|e^{-iH(t_f-t_i)}|\vec{x}, v\rangle_i \\ &= \int[d\vec{x}]\langle v_f|\langle\vec{x}_f|e^{-iH\Delta t}|\vec{x}_{N-1}\rangle|v_{N-1}\rangle\langle v_{N-1}|\langle\vec{x}_{N-1}|e^{-iH\Delta t}|\vec{x}_{N-2}\rangle|v_{N-2}\rangle \\ &\qquad \cdots\langle v_1|\langle\vec{x}_1|e^{-iH\Delta t}|\vec{x}_i\rangle|v_i\rangle \ , \end{aligned} \tag{12}$$

which by use of

$$\langle v_k|\langle\vec{x}_k|e^{-iH\Delta t}|\vec{x}_{k-1}\rangle|v_{k-1}\rangle = \int[d\vec{p}]\langle v_k|e^{i(p_k\frac{\Delta x_k}{\Delta t}-H(x_k))\Delta t}|v_{k-1}\rangle \ , \tag{13}$$

can be written as

$$\langle\vec{x}, v; t_f|\vec{x}, v; t_i\rangle = \int[d\vec{x}][d\vec{p}]e^{i\int(\vec{p}\cdot\dot{\vec{x}}-H(\vec{x}))dt}\langle v_f|v_{N-1}\rangle\langle v_{N-1}|v_{N-2}\rangle\cdots\langle v_1|v_i\rangle \ , \tag{14}$$

where, for example, $[d\vec{x}] \equiv \prod_{i=1}^{3} dx_i\delta(\sum_{j=1}^{3} x_j^2 - r^2)$. Furthermore, if we express

$$\langle v_k|v_{k-1}\rangle \sim 1 - \langle v|\partial_{\vec{x}}|v\rangle\Delta\vec{x} \sim e^{i\omega} \ , \tag{15}$$

we have

$$\langle\vec{x}, v; t_f|\vec{x}, v; t_i\rangle = \int[d\vec{x}][d\vec{p}]e^{i\int(\vec{p}\cdot\dot{\vec{x}}-H(\vec{x}))dt}e^{i\oint\omega} \ , \tag{16}$$

where

$$\omega = i\langle v|\partial_{\vec{x}}|v\rangle\Delta\vec{x} = \frac{-\epsilon_{3ij}}{2r(r+x_3)}x_i\dot{x}_j\Delta t \equiv A_0\Delta t \ . \tag{17}$$

Consequently, to the original action

$$S_0 = \int(\vec{p}\cdot\dot{\vec{x}} - H(\vec{x}))dt \ , \tag{18}$$

we have to add

$$S' = \int A_0 dt \ . \tag{19}$$

Which implies emergence of the geometrical term added to the original Lagrangian and the induced Lagrangian should be

$$L_{S^2} = L_{S^2}^0 + A_0 \ . \tag{20}$$

In order to see the meaning of the induced term, let us consider H-covariant formulation of Lagrangian on the coset space G/H , which in our case is $S^2 = SO(3)/SO(2)$. We can express the original Lagrangian in terms of $SO(3)$ variables $U = u_0 + i\vec{\sigma}\cdot\vec{u}$,

$$L^0_{SO(3)/SO(2)} = r^2 \mathrm{Tr}[DU(DU)^\dagger] \ , \tag{21}$$

where $D \equiv \dfrac{d}{dt} + i\mathcal{A}\dfrac{\sigma_3}{2}$ and $\mathcal{A}$ is a "gauge variable" that compensates the redundant freedom of $SO(2)$ contained in U . We can see the equivalence of this Lagrangian with $L^0_{S^2}$ by integrating out $\mathcal{A}$ in eq.(21), *i.e.*

$$L^0_{SO(3)/SO(2)} \Rightarrow \frac{r^2}{4}\mathrm{Tr}\left[\frac{d}{dt}(U^\dagger\sigma_3 U)\right]^2 = \frac{1}{2}\left(\dot{\vec{x}}\right)^2 = L^0_{S^2} \ , \tag{22}$$

since $U^\dagger\sigma_3 U = \dfrac{\vec{x}}{r}\cdot\vec{\sigma}$ and $\vec{x}^2 = r^2$.

The induced Lagrangian (20) can be written as

$$L_{SO(3)/SO(2)} = L^0_{SO(3)/SO(2)} + k\mathcal{A} \ , \tag{23}$$

in H-covariant form, where $k = -1/2$. Indeed, integrating out $\mathcal{A}$ once again we arrive at

$$L_{SO(3)/SO(2)} \Rightarrow L^0_{S^2} - i\frac{1}{2}\mathrm{Tr}(\dot{U}U^\dagger\sigma_3) - \frac{1}{8r^2} \ , \tag{24}$$

which is equivalent to L_{S^2} (20) up to total divergence and constant term. That is, when the system is described in terms of $SO(3)$ variables, a term proportional to the "gauge variable" $\mathcal{A}$, which was introduced in order to absorb the extra degrees of freedom, is induced. This term, which has topological origin, can be considered as a (0+1)-dimensional Chern-Simons term.

4 S^3 and S^4

The S^3 and S^4 cases go along the same line. In the case of S^4, the quantum constraint can be written as

$$(x_M\gamma^M - r)|v\rangle = 0 \ , \tag{25}$$

where

$$\begin{aligned}
\gamma^M &= (\gamma^1,\gamma^2,\gamma^3,\gamma^4,\gamma^5) \ , \\
\vec{\gamma} &= \begin{pmatrix} 0 & i\vec{\sigma} \\ -i\vec{\sigma} & 0 \end{pmatrix} \ , \ \gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \\
\vec{\gamma} &= (\gamma^1,\gamma^2,\gamma^3) \ .
\end{aligned} \tag{26}$$

Solution of the equation (25) is

$$|v\rangle = \frac{1}{\sqrt{2r(r+x_5)}} \left(\begin{array}{c} r+x_5 \\ x_4 - i\vec{x}\cdot\vec{\sigma} \end{array} \right) , \qquad (27)$$

and the transition amplitude can be written as

$$Z_{S^4} = \int [dx_M][dp_M] \exp\left[i\int \left(p_N \dot{x}_N - H^0_{S^4} \right) dt \right] \mathrm{Tr}\exp\left[\int A^0_{S^4} dt \right] , \qquad (28)$$

where

$$A^0_{S^4} = \frac{-1}{2r(r+x_5)} \sigma_{\mu\nu} x_\mu \dot{x}_\nu \ . \qquad (29)$$

That is, a coupling with the instanton solution is induced. As for the case of S^3, the transition amplitude that follows from the wave function is nothing but eq.(28) with $x_5 = 0$, that is,

$$Z_{S^3} = \int [dx_\mu][dp_\mu] \mathrm{Tr}\exp\left[i\int \left(p_\nu \dot{x}_\nu - H^0_{S^3} \right) dt \right] \mathrm{Tr}\exp\left[\int A^0_{S^3} dt \right] , \qquad (30)$$

the additional term being a coupling with the meron solution[6]

$$A^0_{S^3} = \frac{-1}{2r^2} \sigma_{\mu\nu} x_\mu \dot{x}_\nu \ . \qquad (31)$$

As in the case for S^2 , let us consider H-covariant formulation of the transition amplitude on the coset space G/H , which is $SO(4)/SO(3)$ in the case for S^3 and is $SO(5)/SO(4)$ in the case for S^4.

Let us start with the case of $S^3 = SO(4)/SO(3)$. We write the $SO(4)$ element in the block diagonal 4×4 matrix form

$$SO(4) \ni G_4 = \left(\begin{array}{cc} e^{i\Theta_{\mu\nu}\sigma_{\mu\nu}} & 0 \\ 0 & e^{i\bar{\Theta}_{\mu\nu}\bar{\sigma}_{\mu\nu}} \end{array} \right) \equiv \left(\begin{array}{cc} g & 0 \\ 0 & \bar{g} \end{array} \right) , \qquad (32)$$

where $\sigma_{ij} = \bar{\sigma}_{ji} = \frac{1}{2}\epsilon_{ijk}\sigma_k$, $\sigma_{i4} = -\bar{\sigma}_{i4} = \frac{1}{2}\sigma_i$. The Lagrangian on S^3 can be written in terms of G_4 as

$$L^0_{SO(4)/SO(3)} = \frac{r^2}{2} \mathrm{Tr}(G_4^{-1} D G_4 (G_4^{-1} D G_4)^\dagger) , \qquad (33)$$

where $D \equiv \dfrac{d}{dt} + i\mathcal{A}^{SO(3)}$ and $\mathcal{A}^{SO(3)}$ is the gauge variable that absorbs the extra $SO(3)$ degrees of freedom, which we write as

$$\mathcal{A}^{SO(3)} \equiv \left(\begin{array}{cc} \mathcal{A}_i \frac{\sigma_i}{2} & 0 \\ 0 & \mathcal{A}_i \frac{\sigma_i}{2} \end{array} \right) . \qquad (34)$$

We find that the Lagrangian (33) is equivalent to the naive Lagrangian for a particle on S^3. Indeed, integrating out $\mathcal{A}^{SO(3)}$ we arrive at the following Lagrangian on S^3

$$
\begin{aligned}
L^0_{SO(4)/SO(3)} &\Rightarrow -\frac{r^2}{4}\mathrm{Tr}(g\dot{g}^{-1}-\bar{g}\dot{\bar{g}}^{-1})^2 \\
&= \frac{r^2}{4}\mathrm{Tr}\left[\frac{d}{dt}(\bar{g}^{-1}g)\frac{d}{dt}(g^{-1}\bar{g})\right] \\
&= \frac{r^2}{4}\mathrm{Tr}\dot{Q}_3\dot{Q}_3^{-1} = \frac{1}{2}\dot{x}_\mu^2 = L^0_{S^3}\ ,
\end{aligned}
\tag{35}
$$

where $Q_3 \equiv \bar{g}^{-1}g = \frac{x_4}{r} + i\frac{x_i}{r}\sigma_i$ and $\sum_{\mu=1}^{4} x_\mu^2 = r^2$.

Next we add a term $\mathrm{Tr}(K\mathcal{A}^{SO(3)})$ to the Lagrangian, where the constant K is given by

$$
K = \begin{pmatrix} k_i\sigma_i & 0 \\ 0 & k_i\sigma_i \end{pmatrix} . \tag{36}
$$

Then, integrating out $\mathcal{A}^{SO(3)}$, we find that the Lagrangian,

$$
L_{SO(4)/SO(3)} = L^0_{SO(4)/SO(3)} + \mathrm{Tr}(K\mathcal{A}^{SO(3)})\ , \tag{37}
$$

goes to

$$
L_{SO(4)/SO(3)} \Rightarrow L^0_{S^3} - i\mathrm{Tr}\left[G_4\dot{G}_4^{-1}K\right] - \frac{1}{2r^2}\mathrm{Tr}(K^2)\ . \tag{38}
$$

Furthermore, we can show that the transition amplitude $Z_{SO(4)/SO(3)}$ corresponding to this Lagrangian is identical to Z_{S^3}. As we can write

$$
G_4 = h^{-1}\begin{pmatrix} \bar{g}^{-1}g & 0 \\ 0 & 1 \end{pmatrix}\ , \tag{39}
$$

we have

$$
G_4\dot{G}_4^{-1} = h^{-1}(-4iA^0_{S^3})h + h^{-1}\dot{h}\ , \tag{40}
$$

where h is an $SO(3)$ element written in 4×4 block diagonal matrix form. Thus the Lagrangian is expressed as

$$
L_{SO(4)/SO(3)} \Rightarrow L^0_{S^3} - i\mathrm{Tr}(Kh^{-1}\dot{h}) - 4\mathrm{Tr}(hKh^{-1}A^0_{S^3})\ + \text{const.}\ . \tag{41}
$$

We note that this Lagrangian has the same form as that obtained in ref.[1]. Here if we define

$$
\begin{pmatrix} S^i\sigma_i & 0 \\ 0 & S^i\sigma_i \end{pmatrix} \equiv \frac{1}{2}h^{-1}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} h\ , \tag{42}
$$

the corresponding Hamiltonian can be written as

$$H = H^0_{S^3} - 2S^i (A^0_{S^3})^i \ , \tag{43}$$

where S^i is a spin variable that satisfies $[S^i, S^j] = i\epsilon^{ijk} S^k$. Based on this Hamiltonian with $k_i = -\frac{1}{4}\delta_{i3}$, we derive the transition amplitude by integrating the spin degrees of freedom[7][8],

$$Z_{SO(4)/SO(3)} = \int [dx_\mu][dp_\mu] \exp\left[i \int \left(p_\nu \dot{x}_\nu - H^0_{S^3} \right) dt \right] \mathrm{Tr} \exp\left[i \int A^0_{S^3} dt \right] \ . \tag{44}$$

We have confirmed this equation under the gauge condition $(A^0_{S^3})^1 = (A^0_{S^3})^2 = 0$. This expression coincides completely with the previous Z_{S^3} .

Next we turn to the discussion on S^4. The naive Lagrangian for the particle on S^4 in terms of $SO(5)$ variable G_5 can be written as

$$L^0_{SO(5)/SO(4)} = \frac{r^2}{2} \mathrm{Tr} \left(G_5^{-1} D G_5 \left(G_5^{-1} D G_5 \right)^\dagger \right) \ , \tag{45}$$

where $D \equiv \dfrac{d}{dt} + i\mathcal{A}^{SO(4)}$ is the $SO(4)$-covariant derivative. By integrating out $\mathcal{A}^{SO(4)}$, this Lagrangian can be seen to be equivalent to $L^0_{S^4}$. This may be explicitly shown in the representation such as

$$\mathcal{A}^{SO(4)} = \begin{pmatrix} \mathcal{A}_{\mu\nu}\sigma_{\mu\nu} & 0 \\ 0 & \mathcal{A}_{\mu\nu}\bar{\sigma}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_+ & 0 \\ 0 & \mathcal{A}_- \end{pmatrix} \equiv \mathcal{A}^{SO(4)}_a T_a \ , \tag{46}$$

where T_a is the generator corresponding to the $SO(4)$ subgroup of $SO(5)$. We separate as $G_5 \dot{G}_5^{-1} \equiv i g^a_\perp T_a + i g^\alpha_\parallel T_\alpha \equiv G_\perp + G_\parallel$ (T_α is the remaining generator of $SO(5)$). Integrating out $\mathcal{A}^{SO(4)}$, we see

$$L^0_{SO(5)/SO(4)} \Rightarrow -\frac{r^2}{2} \mathrm{Tr} G_\parallel^2 = \frac{1}{2} \dot{x}_M^2 = L^0_{S^4} \ , \tag{47}$$

where $G_5^{-1} \gamma^5 G_5 = \sum\limits_{M=1}^{5} \dfrac{x_M}{r} \gamma_M$ and $\sum\limits_{M=1}^{5} x_M^2 = r^2$.

Next we assume that the term $\mathrm{Tr}(K\mathcal{A}^{SO(4)})$ has been induced to the system on S^4 , where the constant K is the algebra of $SO(4)$, the general form of which is given by

$$K = \begin{pmatrix} K_+ & 0 \\ 0 & K_- \end{pmatrix} = K_a T_a \ . \tag{48}$$

Then integrating out $\mathcal{A}^{SO(4)}$ we obtain the Lagrangian on S^4

$$L_{SO(5)/SO(4)} \Rightarrow L^0_{S^4} - i \mathrm{Tr}(G_5 \dot{G}_5^{-1} K) - \frac{1}{2r^2} \mathrm{Tr}(K^2) \ . \tag{49}$$

As in the case of S^3, we calculate transition amplitude by means of the Hamiltonian

$$H = H^0_{S^4} + 2S^i (A^0_{S^4})^i \ , \tag{50}$$

which corresponds to the effective Lagrangian (49), and integrate out the spin variables S^i defined by

$$\begin{pmatrix} 0 & 0 \\ 0 & S^i\sigma_i \end{pmatrix} \equiv \frac{1}{2}\tilde{h}^{-1}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_3 \end{pmatrix}\tilde{h} \ , \tag{51}$$

where $\tilde{h}$ is an element of the $SO(4)$ subgroup. Then, we arrive at

$$Z_{SO(5)/SO(4)} = \int [dx_M][dp_M] \exp\left[i\int \left(p_N\dot{x}_N - H^0_{S^4}\right) dt\right] \mathrm{Tr}\exp\left[i\int A^0_{S^4} dt\right] \ , \tag{52}$$

where the choice has been used of $K_+ = 0$, $K_- = -\frac{1}{2}\sigma_3$.

Thus, we see that for both cases S^3 and S^4, the (0+1)-dimensional Chern-Simons terms, that are proportional to the gauge variable, are induced in the course of quantization.

5 S^n

What has been argued so far can be generalized to all n . First, we obtain the transition amplitude for a particle on S^n that follows from the wave function with the "on sphere condition" taken into account. In order to do this, we consider the cases $n = 2m$ and $n = 2m-1$ separately. We start with $n = 2m$ case and prepare $2^m \times 2^m$ matrices $\Gamma^{(2m)}_N$ $(N = 1, 2, \cdots, 2m+1)$

$$\{\Gamma^{(2m)}_M, \Gamma^{(2m)}_N\} = 2\delta_{MN}, \qquad [\Gamma^{(2m)}_M, \Gamma^{(2m)}_N] = 2i\Sigma^{(2m)}_{MN} \ . \tag{53}$$

The explicit form of $\Gamma^{(2m)}_N$ can be obtained as

$$\begin{aligned} \Gamma^{(2m)}_i &= \sigma_2 \otimes \Gamma^{(2m-2)}_i \qquad (i = 1, \cdots, 2m-1) \ , \\ \Gamma^{(2m)}_{2m} &= \sigma_1 \otimes 1 \ , \\ \Gamma^{(2m)}_{2m+1} &= \sigma_3 \otimes 1 \ , \end{aligned} \tag{54}$$

and $SO(2m)$ generators $\Sigma^{(2m)}_{\mu\nu}$ $(\mu, \nu = 1, 2, \cdots, 2m)$ are expressed in block diagonal form as

$$\begin{pmatrix} \Sigma^{(2m)+}_{\mu\nu} & 0 \\ 0 & \Sigma^{(2m)-}_{\mu\nu} \end{pmatrix} \ . \tag{55}$$

The spinor wave function $|v\rangle$ in $2^m \times 2^{m-1}$ matrix form is constrained to satisfy the "on sphere condition"

$$(x_N \Gamma_N^{(2m)} - r)|v\rangle = 0 \; . \tag{56}$$

The non-trivial solution to the equation is given as

$$|v\rangle = \frac{1}{\sqrt{2r(r+x_{2m+1})}} \begin{pmatrix} r + x_{2m+1} \\ x_{2m} - i x_i \Gamma_i^{(2m-2)} \end{pmatrix} \; . \tag{57}$$

With the help of $|v\rangle$ we perform the path integral to obtain the transition amplitude,

$$Z_{S^{2m}} = \int [dx_M][dp_M] \exp\left[i \int (p_N \dot{x}_N - H^0_{S^{2m}}) dt \right] \mathrm{Tr} \exp\left[i \int A^0_{S^{2m}} dt \right] \; , \tag{58}$$

where

$$A^0_{S^{2m}} = \frac{-1}{2r(r+x_{2m+1})} \Sigma^{(2m)+}_{\mu\nu} x_\mu \dot{x}_\nu \; . \tag{59}$$

Thus, we can claim that a coupling to the generalized instanton configuration is induced [5].

The transition amplitude for a particle on S^{2m-1} is nothing but eq.(58) with $x_{2m+1} = 0$. That is,

$$Z_{S^{2m-1}} = \int [dx_\mu][dp_\mu] \exp\left[i \int (p_\nu \dot{x}_\nu - H^0_{S^{2m-1}}) dt \right] \mathrm{Tr} \exp\left[i \int A^0_{S^{2m-1}} dt \right] \; , \tag{60}$$

The additional term is a coupling to the generalized meron solution in arbitrary odd dimensions[6],

$$A^0_{S^{2m-1}} = \frac{-1}{2r^2} \Sigma^{(2m)+}_{\mu\nu} x_\mu \dot{x}_\nu \; . \tag{61}$$

Next, exactly as in the previous discussions, having in mind that $S^n = SO(n+1)/SO(n)$ we can describe the system using the elements of $SO(n+1)$. This suggest that the induced term, which was obtained through the above mentioned path integration, appears as a term proportional to the "gauge variable" $\mathcal{A}^{SO(n)}$ that was introduced in order to absorb the extra degrees of freedom. Thus, for the description of quantum mechanics on S^n using the $SO(n+1)$ variables, the gauge variable $\mathcal{A}^{SO(n)}$ is expected to be induced. Namely, we claim that the induced term is a term proportional to the "gauge variable" also for the general dimension n.

6 Discussions

We have seen that, when describing quantum mechanics on the sphere in terms of the wave functions that satisfy the "square root" of "on sphere condition", a particular gauge configuration appears in the transition amplitude due to the geometrical reasons. These results are consistent with those obtained in ref.[9] from a different point of view. These configurations are monopoles and (generalized) instantons for even dimensional spheres and (generalized) merons for odd dimensional spheres.

Furthermore, we have shown in this note that it is possible to interpret this situation as an induction of a term proportional to the "gauge variable" that was introduced in order to absorb the extra degrees of freedom, when we describe the S^n system in terms of the $SO(n+1)$ variables according to the relation $S^n = SO(n+1)/SO(n)$. The induced terms have topological meaning and can be considered as a Chern-Simons terms in $0+1$ dimensions.

If we extend this result, obtained in quantum mechanics, to the field theories where the fields are constrained to the sphere, we could expect an induction of Chern-Simons gauge fields. For example, for $O(3)$ sigma model in $2+1$ dimensions, fields are on $S^2 = SU(2)/U(1)$ and we expect $U(1)$ Chern-Simons term to be induced in this case. This possibility was also suggested in refs.[10],[11].

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